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Free energy and Green function for the multi-coupled spin system

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Abstract

The free energy and Green function of the multi-coupled spin chain in low-energy theory are given by using the functional bosonization method. We deduce that 2^m coupled spinless chains ($m = 0, 1, 2, \ldots$), by using a series of unitary transformations, can be reduced to some uncoupled chains, but for other numbers of coupled spinless chains we need to solve a series of inverse matrices generally; for either case, the system can be formally separated into the charge and spin parts.

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1. Introduction

Among a few non-perturbation theories dealing with the one-dimensional Fermi system, bosonization is one of the successful methods [1, 2]. Traditional bosonization is the operator method which directly represents the fermion field in terms of the density operator [3, 4] in the low-energy scales. The Luttinger liquid and Tomonaga–Luttinger liquid model are two typical solvable ones. Due to the strong correlated effect, their characteristics cannot be interpreted with Landau Fermi liquid theory [5–8]. Functional bosonization is another effective non-perturbation method, of which the idea is based on functional integration. In the traditional bosonization method, Klein factors must be introduced in order to manifest the Fermi anticommutations. Although these factors do not influence the Green function for the single chain, we must be careful in the multi-chain system. The merit of the functional bosonization is refraining from the introduction of these factors. This method has been applied to the one-dimensional Tomonaga–Luttinger model with forward scattering [9]. In an electron–phonon coupling one-dimensional system, the functional-integral approach is extensively adopted to reveal Peierls dimerization [10–12].

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The one-dimensional chain and spin ladder models are successful for bosonization [13, 14]. Recently the low-energy theory in multiwall carbon nanotubes was analysed by the bosonization method and characteristic Luttinger liquid power laws were given [15] by considering intershell electron tunnelling and the long-ranged Coulomb effect. The multi-coupled chain is therefore important in nature. Our motivation is to study the multi-coupled spin chain system. Based on the functional bosonization, the free energy and Green function are given. We deduce from the conclusion of the coupled spin chain system that 2^m coupled spinless chains with forward scattering can be decoupled to a series of spinless independent Tomonaga models, but other numbers of coupled chains do not have this characteristic and we need to solve the eigenvalues for a large matrix.

The paper is arranged as follows. In section 2, we use functional bosonization on the multi-coupled spin chain system to give the expression for the fermion system with the boson operators. The free energy and Green function are given in sections 3 and 4 respectively. In sections 5 and 6, the two-coupled spin chain system, as a example, is thoroughly solved; we deduce that 2^m coupled spinless chains may be decoupled to some independent spinless chain by a series of unitary transformations.

2. Functional bosonization for the multi-coupled spin system

The construction of the low-energy theory is based on the expansion of the electron operator for spin $\sigma = \uparrow, \downarrow$ on shell $n = 1, 2, \ldots, N$. The electron field may be expressed as the slow-variation part of ψ_{R} near the right-hand Fermi and ψ_{L} near the left-hand Fermi point. In the low-energy scale, the Hamiltonian describing this system is given as

$$H = -\mathrm{i}v_{\mathrm{F}} \int \mathrm{d}x \sum_{n,\sigma} [\psi^{+}_{\mathrm{R}n\sigma}(x)\partial_x \psi_{\mathrm{R}n\sigma}(x) - \psi^{+}_{\mathrm{L}n\sigma}(x)\partial_x \psi_{\mathrm{L}n\sigma}(x)] + \int \mathrm{d}x\, \mathrm{d}y\, \rho^{\mathrm{T}}(x) V(x-y)\rho(y) \tag{1}$$

with the vector of the fluctuation density

$$\rho^{\mathrm{T}}(x) = (\rho_{\mathrm{R}1\uparrow}(x), \rho_{\mathrm{R}1\downarrow}(x), \ldots, \rho_{\mathrm{R}N\uparrow}(x), \rho_{\mathrm{R}N\downarrow}(x), \rho_{\mathrm{L}1\uparrow}(x), \rho_{\mathrm{L}1\downarrow}(x), \ldots, \rho_{\mathrm{L}N\uparrow}(x), \rho_{\mathrm{L}N\downarrow}(x))$$

and $\rho_{\mathrm{R/L}i\sigma}(x) = \psi^{+}_{\mathrm{R/L}i\sigma}(x)\psi_{\mathrm{R/L}i\sigma}(x)$ the fluctuation density. We have chosen $\hbar = 1$ and neglected the back-scattering and the umklapp processes, since these processes need large enough momentum ($\sim 2p_f$ and $\sim 4p_f$). The interaction potential is chosen as the local interaction $V(x-y) = \boldsymbol{V}\delta(x-y)$, where the $4N \times 4N$ matrix $\boldsymbol{V}$ is written as

$$\boldsymbol{V} = \begin{pmatrix}
V_{11\mathrm{RRp}} & V_{11\mathrm{RRv}} & \cdots & V_{1N\mathrm{RRp}} & V_{1N\mathrm{RRv}} & V_{11\mathrm{RLp}} & V_{11\mathrm{RLv}} & \cdots & V_{1N\mathrm{RLp}} & V_{1N\mathrm{RLv}} \\
V_{11\mathrm{RRv}} & V_{11\mathrm{RRp}} & \cdots & V_{1N\mathrm{RRv}} & V_{1N\mathrm{RRp}} & V_{11\mathrm{RLv}} & V_{11\mathrm{RLp}} & \cdots & V_{1N\mathrm{RLv}} & V_{1N\mathrm{RLp}} \\
V_{21\mathrm{RRp}} & V_{21\mathrm{RRv}} & \cdots & V_{2N\mathrm{RRp}} & V_{2N\mathrm{RRv}} & V_{21\mathrm{RLp}} & V_{21\mathrm{RLv}} & \cdots & V_{2N\mathrm{RLp}} & V_{2N\mathrm{RLv}} \\
V_{21\mathrm{RRv}} & V_{21\mathrm{RRp}} & \cdots & V_{2N\mathrm{RRv}} & V_{2N\mathrm{RRp}} & V_{21\mathrm{RLv}} & V_{21\mathrm{RLp}} & \cdots & V_{2N\mathrm{RLv}} & V_{2N\mathrm{RLp}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
V_{N1\mathrm{RRp}} & V_{N1\mathrm{RRv}} & \cdots & V_{NN\mathrm{RRp}} & V_{NN\mathrm{RRv}} & V_{N1\mathrm{RLp}} & V_{N1\mathrm{RLv}} & \cdots & V_{NN\mathrm{RLp}} & V_{NN\mathrm{RLv}} \\
V_{N1\mathrm{RRv}} & V_{N1\mathrm{RRp}} & \cdots & V_{NN\mathrm{RRv}} & V_{NN\mathrm{RRp}} & V_{N1\mathrm{RLv}} & V_{N1\mathrm{RLp}} & \cdots & V_{NN\mathrm{RLv}} & V_{NN\mathrm{RLp}} \\
V_{11\mathrm{RLp}} & V_{11\mathrm{RLv}} & \cdots & V_{1N\mathrm{RLp}} & V_{1N\mathrm{RLv}} & V_{11\mathrm{LLp}} & V_{11\mathrm{LLv}} & \cdots & V_{1N\mathrm{LLp}} & V_{1N\mathrm{LLv}} \\
V_{11\mathrm{RLv}} & V_{11\mathrm{RLp}} & \cdots & V_{1N\mathrm{RLv}} & V_{1N\mathrm{RLp}} & V_{11\mathrm{LLv}} & V_{11\mathrm{LLp}} & \cdots & V_{1N\mathrm{LLv}} & V_{1N\mathrm{LLp}} \\
V_{21\mathrm{RLp}} & V_{21\mathrm{RLv}} & \cdots & V_{2N\mathrm{RLp}} & V_{2N\mathrm{RLv}} & V_{21\mathrm{LLp}} & V_{21\mathrm{LLv}} & \cdots & V_{2N\mathrm{LLp}} & V_{2N\mathrm{LLv}} \\
V_{21\mathrm{RLv}} & V_{21\mathrm{RLp}} & \cdots & V_{2N\mathrm{RLv}} & V_{2N\mathrm{RLp}} & V_{21\mathrm{LLv}} & V_{21\mathrm{LLp}} & \cdots & V_{2N\mathrm{LLv}} & V_{2N\mathrm{LLp}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
V_{N1\mathrm{RLp}} & V_{N1\mathrm{RLv}} & \cdots & V_{NN\mathrm{RLp}} & V_{NN\mathrm{RLv}} & V_{N1\mathrm{LLp}} & V_{N1\mathrm{LLv}} & \cdots & V_{NN\mathrm{LLp}} & V_{NN\mathrm{LLv}} \\
V_{N1\mathrm{RLv}} & V_{N1\mathrm{RLp}} & \cdots & V_{NN\mathrm{RLv}} & V_{NN\mathrm{RLp}} & V_{N1\mathrm{LLv}} & V_{N1\mathrm{LLp}} & \cdots & V_{NN\mathrm{LLv}} & V_{NN\mathrm{LLp}}
\end{pmatrix}. \tag{2}$$

Due to the absence of the magnetic interaction, we have used the indices p and v to represent the parallel and vertical spin of two electrons respectively, i.e. $(\uparrow\uparrow, \downarrow\downarrow)$ and $(\uparrow\downarrow, \downarrow\uparrow)$. The

physical character demands that the matrix $\boldsymbol{V}$ is a symmetric one, which has properties

$$V_{ij\mathrm{RRp}} = V^T_{ji\mathrm{RRp}} \qquad V_{ij\mathrm{RRv}} = V^T_{ji\mathrm{RRv}}$$
$$V_{ij\mathrm{RLp}} = V^T_{ji\mathrm{RLp}} = V_{ij\mathrm{LRp}} \qquad V_{ij\mathrm{RLv}} = V^T_{ji\mathrm{RLv}} = V_{ij\mathrm{LRv}}.$$

The regularized partition function of the system is given by the formal ratio:

$$\frac{Z}{Z_0} = \frac{\int [D\bar{\Psi}][D\Psi] \exp(\mathrm{i}S[\bar{\Psi}, \Psi, V])}{\int [D\bar{\Psi}][D\Psi] \exp(\mathrm{i}S[\bar{\Psi}, \Psi, 0])} \tag{3}$$

where $[D\bar{\Psi}] = \prod_{i=1,\sigma}^{N} \mathrm{d}\bar{\psi}_{\mathrm{R}i\sigma}\, \mathrm{d}\bar{\psi}_{\mathrm{L}i\sigma}$ and $[D\Psi] = \prod_{i=1,\sigma}^{N} \mathrm{d}\psi_{\mathrm{R}i\sigma}\, \mathrm{d}\psi_{\mathrm{L}i\sigma}$ are the integration measure for the anticommuting fermion field. $S[\bar{\Psi}, \Psi, V]$ is the action for the system and so is $S[\bar{\Psi}, \Psi, 0]$ in the absence of the Coulomb interaction V. The action of the multi-coupled spin system is given by

$$S[\bar{\Psi}, \Psi, V] = \int \mathrm{d}t\, \mathrm{d}x \sum_{n\sigma} [\bar{\psi}_{\mathrm{R}n\sigma}(x,t) D_{\mathrm{R}} \psi_{\mathrm{R}n\sigma}(x,t) + \bar{\psi}_{\mathrm{L}n\sigma}(x,t) D_{\mathrm{L}} \psi_{\mathrm{L}n\sigma}(x,t)]$$
$$- \int \mathrm{d}t\, \mathrm{d}x\, \rho^{\mathrm{T}}(x,t) \boldsymbol{V} \rho(x,t) \tag{4}$$

where $D_{\mathrm{R}} = \mathrm{i}(\partial_t + v_{\mathrm{F}}\partial_x)$ and $D_{\mathrm{L}} = \mathrm{i}(\partial_t - v_{\mathrm{F}}\partial_x)$. Due to the four-fermion interaction in the system, the partition function becomes after introducing $4N$ Hubbard–Stratonorich boson fields $\phi_{\mathrm{R/L}n\sigma}(x,t)$,

$$\frac{Z}{Z_0} = \frac{\int [D\bar{\Psi}][D\Psi][D\rho][D\phi] \exp(\mathrm{i}S[\bar{\Psi}, \Psi, \phi, \rho, V])}{\int [D\bar{\Psi}][D\Psi][D\rho]D[\phi] \exp(\mathrm{i}S[\bar{\Psi}, \Psi, \phi, \rho, 0])} \tag{5}$$

where $[D\phi] = \prod_{i=1,\sigma}^{N} \mathrm{d}\phi_{\mathrm{R}i\sigma}\, \mathrm{d}\phi_{\mathrm{L}i\sigma}$ and $[D\rho] = \prod_{i=1,\sigma}^{N} \mathrm{d}\rho_{\mathrm{R}i\sigma}\, \mathrm{d}\rho_{\mathrm{L}i\sigma}$. The action in equation (5) should be

$$S[\bar{\Psi}, \Psi, \phi, \rho, V] = \int \mathrm{d}t\, \mathrm{d}x \sum_{n\sigma} [\bar{\psi}_{\mathrm{R}n\sigma}(x,t)(D_{\mathrm{R}} - \phi_{\mathrm{R}n\sigma})\psi_{\mathrm{R}n\sigma}(x,t)$$
$$+ \bar{\psi}_{\mathrm{L}n\sigma}(x,t)(D_{\mathrm{L}} - \phi_{\mathrm{L}n\sigma})\psi_{\mathrm{L}n\sigma}(x,t)]$$
$$+ \int \mathrm{d}t\, \mathrm{d}x\, \phi^{\mathrm{T}}(x,t)\rho(x,t) - \int \mathrm{d}t\, \mathrm{d}x\, \rho^{\mathrm{T}}(x,t)\boldsymbol{V}\rho(x,t). \tag{6}$$

Corresponding to the vector ρ^{T}, we have denoted the vector $\phi^{\mathrm{T}}(x,t)$ as

$$\phi^{\mathrm{T}}(x,t) = (\phi_{\mathrm{R}1\uparrow}(x,t), \phi_{\mathrm{R}1\downarrow}(x,t), \phi_{\mathrm{R}2\uparrow}(x,t), \phi_{\mathrm{R}2\downarrow}(x,t), \ldots, \phi_{\mathrm{R}N\uparrow}(x,t), \phi_{\mathrm{R}N\downarrow}(x,t),$$
$$\phi_{\mathrm{L}1\uparrow}(x,t), \phi_{\mathrm{L}1\downarrow}(x,t), \ldots, \phi_{\mathrm{L}N\uparrow}(x,t), \phi_{\mathrm{L}N\downarrow}(x,t)).$$

It is seen that the introduction of the auxillary fields $\phi_{\mathrm{R/L}i\sigma}$ makes the density $\rho_{\mathrm{R/L}i\sigma}$ become a series of free fields. Due to the introduction of $4N$ Hubbard–Stratonorich boson fields $\phi_{\mathrm{R/L}n\sigma}(x,t)$, we can integrate out electron fields from equation (5). After this, the partition function becomes

$$\frac{Z}{Z_0} = \frac{\int [D\rho][D\phi] \exp(\mathrm{i}S_{\mathrm{eff}}(\phi, \rho, V])}{\int [D\rho][D\phi] \exp(\mathrm{i}S_{\mathrm{eff}}[\phi, \rho, 0])} \tag{7}$$

where the effective action is

$$S_{\mathrm{eff}}[\phi, \rho, V] = -\mathrm{i}T_r \ln \hat{M} + \int \mathrm{d}t\, \mathrm{d}x\, \phi^{\mathrm{T}}(x,t)\rho(x,t) - \int \mathrm{d}t\, \mathrm{d}x\, \rho^{\mathrm{T}}(x,t)\boldsymbol{V}\rho(x,t). \tag{8}$$

The matrix $\hat{M}$ is denoted as

$$\hat{M} = \hat{M}_0 + \hat{\Phi} \tag{9}$$

with

$$\hat{M}_0 = \begin{pmatrix} D_{\mathrm{R}} & 0 \\ 0 & D_{\mathrm{L}} \end{pmatrix} \bigotimes \begin{pmatrix} I_{2N\times 2N} & 0 \\ 0 & I_{2N\times 2N} \end{pmatrix}$$

$I_{2N\times 2N}$ is the $2N \times 2N$ unit matrix. The $4N \times 4N$ order matrix $\hat{\Phi}$ is a diagonal one, in which the diagonal elements are in proper order

$$\phi_{\mathrm{R}1\uparrow}(x,t), \phi_{\mathrm{R}1\downarrow}(x,t), \phi_{\mathrm{R}2\uparrow}(x,t), \phi_{\mathrm{R}2\downarrow}(x,t), \ldots, \phi_{\mathrm{R}N\uparrow}(x,t), \phi_{\mathrm{R}N\downarrow}(x,t),$$
$$\phi_{\mathrm{L}1\uparrow}(x,t), \phi_{\mathrm{L}1\downarrow}(x,t), \ldots, \phi_{\mathrm{L}N\uparrow}(x,t), \phi_{\mathrm{L}N\downarrow}(x,t)$$

respectively. Using the formula $\det A = \exp(T_r \ln A)$ and $\ln(A+B) = \ln A + B\int_0^1 \mathrm{d}\lambda\,(A+\lambda B)^{-1}$, we have

$$T_r \ln \hat{M} = T_r \ln \hat{M}_0 + \int_0^1 \mathrm{d}\lambda \int \mathrm{d}t\,\mathrm{d}x\, T_r \Phi(x,t)\tilde{G}(x,t,x',t',[\lambda\phi])\mid_{t'\to t\ \ x'\to x} \tag{10}$$

where the Green function in equation (10) satisfies the following equation:

$$\hat{M}(x,t)\tilde{G}(x,t,x',t',[\phi]) = \delta(x-x')\delta(t-t'). \tag{11}$$

Since the matrix $\hat{M}$ is diagonal, so is the Green function matrix $\tilde{G}(x,t,x',t',[\phi])$. The matrix elements of the Green function satisfy

$$\begin{aligned} (D_{\mathrm{R}} - \phi_{\mathrm{R}i\sigma}(x,t))\tilde{G}_{\mathrm{R}i\sigma}(x,t,x',t',[\phi]) &= \delta(x-x')\delta(t-t') \\ (D_{\mathrm{L}} - \phi_{\mathrm{L}i\sigma}(x,t))\tilde{G}_{\mathrm{L}i\sigma}(x,t,x',t',[\phi]) &= \delta(x-x')\delta(t-t'). \end{aligned} \tag{12}$$

In order to solve equation (12) we make an ansatz

$$\tilde{G}_{\mathrm{R/L}i\sigma}(x,t,x',t',[\phi]) = \tilde{G}^0_{\mathrm{R/L}i\sigma}(x,t,x',t')\exp[Q_{\mathrm{R/L}i\sigma}(x,t) - Q_{\mathrm{R/L}i\sigma}(x',t')] \tag{13}$$

where we have introduced $4N$ functions $Q_{\mathrm{R/L}i\sigma}(x,t)$. Green functions $\tilde{G}^0_{\mathrm{R/L}i\sigma}(x,t,x',t')$ satisfy

$$D_{\mathrm{R/L}}\tilde{G}^0_{\mathrm{R/L}i\sigma}(x,t,x',t') = \delta(x-x')\delta(t-t'). \tag{14}$$

It is seen that the Green functions $\tilde{G}^0_{\mathrm{R/L}i\sigma}(x,t,x',t')$ do not depend on the index (i,σ) because there is no magnetic interaction in the free system, so we denote them as $\tilde{G}^0_{\mathrm{R/L}}(x,t,x',t')$ in the following. The solution for equation (14) is

$$\tilde{G}^0_{\mathrm{R/L}}(x,t,x',t') = \mp\frac{\exp(\pm \mathrm{i}p_{\mathrm{F}}(x-x'))}{2\pi}\frac{1}{(x-x') \mp v_{\mathrm{F}}(t-t')}. \tag{15}$$

After substituting equation (13) into (12), we have that $4N \times 4N$ functions satisfy

$$\mathrm{i}(\partial_t \pm v_{\mathrm{F}}\partial_x)Q_{\mathrm{R/L}i\sigma}(x,t) = \phi_{\mathrm{R/L}i\sigma}(x,t). \tag{16}$$

In the momentum space, the solutions for the above equation are

$$Q_{\mathrm{R/L}i\sigma}(x,t) = \frac{1}{TL}\sum_{q,\omega}\frac{1}{\omega \mp v_{\mathrm{F}}q}\phi_{\mathrm{R/L}i\sigma}(q,\omega)\exp[\mathrm{i}(qx-\omega t)]. \tag{17}$$

Equation (15) shows that there is a divergence point in Green function $\tilde{G}^0_{\mathrm{R/L}}(x,t)$. Since $\tilde{G}^0_{\mathrm{R/L}}(x,t)$ and $\tilde{G}_{\mathrm{R/L}}(x,t,x',t',[\phi])$ differing only in phase, this divergence also exists in $G_{\mathrm{R/L}}(x,t,x',t',[\phi])$. We adopt the point splitting method [16] to regularize for the functional Green function,

$$\tilde{G}_{\mathrm{R/L}}(x,t,x',t',[\phi])\mid_{x'\to x,t'\to t}$$
$$= \tfrac{1}{2}\lim_{\eta\to 0^+}[\tilde{G}_{\mathrm{R/L}}(x,t,x+\eta,t,[\phi]) + \tilde{G}_{\mathrm{R/L}}(x,t,x-\eta,t,[\phi])]. \tag{18}$$

Substituting equations (13) and (17) into (18), we have the concrete form of the Green functions,

$$\tilde{G}_{\mathrm{R/L}i\sigma}(x,t,x',t',[\phi])|_{x'\to x,t'\to t} = \mp\frac{\mathrm{i}\lambda}{2\pi TL}\sum_{q,\omega}\frac{q}{\omega\mp v_{\mathrm{F}}q}\phi_{\mathrm{R/L}i\sigma}(q,\omega)\exp[\mathrm{i}(qx-\omega t)]. \tag{19}$$

Therefore the effective action (8) can be written as

$$\begin{aligned}S_{\mathrm{eff}}[\phi,\rho,V] = &-\frac{1}{4\pi TL}\sum_{q,\omega}\sum_{i,\sigma}\left[\frac{q}{v_{\mathrm{F}}q-\omega}\phi_{\mathrm{R}i\sigma}(-q,-\omega)\phi_{\mathrm{R}i\sigma}(q,\omega)\right.\\ &\left.+\frac{q}{v_{\mathrm{F}}q+\omega}\phi_{\mathrm{L}i\sigma}(-q,-\omega)\phi_{\mathrm{L}i\sigma}(q,\omega)\right]\\ &-\frac{1}{TL}\sum_{q,\omega}\phi^{\mathrm{T}}(-q,-\omega)\rho(q,\omega)-\frac{1}{TL}\sum_{q,\omega}\rho^{\mathrm{T}}(-q,-\omega)\boldsymbol{V}\rho(q,\omega).\end{aligned} \tag{20}$$

Equation (20) shows that the system described by the fermion field has been turned into the representation in terms of the density bosonic field and the auxillary bosonic fields.

3. Free energy for the multi-coupling spin chain

It is seen that the effective action is the quadratic function for the boson fields ρ and ϕ. We can integrate either of them. In order to give the free energy of the system, we integrate out the boson fields ϕ first. After this we have

$$\begin{aligned}S_{\mathrm{eff}}[\rho,V] = &\frac{1}{TL}\sum_{q,\omega}\sum_{i,\sigma}[F_{\mathrm{R}}\rho_{\mathrm{R}i\sigma}(-q,-\omega)\rho_{\mathrm{R}i\sigma}(q,\omega)+F_{\mathrm{L}}\rho_{\mathrm{L}i\sigma}(-q,-\omega)\rho_{\mathrm{L}i\sigma}(q,\omega)]\\ &-\frac{1}{TL}\sum_{q,\omega}\rho^{\mathrm{T}}(-q,-\omega)\boldsymbol{V}\rho(q,\omega)\end{aligned} \tag{21}$$

with $F_{\mathrm{R/L}} = \pi(v_{\mathrm{F}}q\mp\omega)/q$. Considering the properties of the interacting matrix $\boldsymbol{V}$, we make a unitary transformation

$$\rho'(x) = \boldsymbol{U}\rho(x) \tag{22}$$

where the new vector is described as

$$\rho'^{\mathrm{T}}(x) = (\rho_{1\mathrm{R}}(x),\rho_{1\mathrm{L}}(x),\ldots,\rho_{N\mathrm{R}}(x),\rho_{N\mathrm{L}}(x),\chi_{1\mathrm{R}}(x),\chi_{1\mathrm{L}}(x),\ldots,\chi_{N\mathrm{R}}(x),\chi_{N\mathrm{L}}(x))$$

and the transformation is chosen as

$$U = \frac{1}{\sqrt{2}}\begin{array}{c}\begin{array}{cccc}2i-1 & 2i & 2N+2i-1 & 2N+2i\end{array}\\ \begin{pmatrix}1 & 1 & & \\ & & 1 & 1\\ 1 & -1 & & \\ & & 1 & -1\end{pmatrix}\end{array}\begin{array}{ll}\cdots & 2i-1\text{ array}\\ \cdots & 2i\text{ array}\\ \cdots & 2N+2i-1\text{ array}\\ \cdots & 2N+2i\text{ array}\end{array} \tag{23}$$

where $i = 1,2,\ldots,N$. In the transformation matrix (23), we only write the non-zero matrix elements and give their sites. Observing equations (22) and (23), $\rho_{i\mathrm{R/L}}$ and $\chi_{i\mathrm{R/L}}$ represent the density of charge and of spin in the ith chain respectively. It is proved that the potential becomes a diagonal matrix after the transformation

$$\boldsymbol{K}'_U \equiv \boldsymbol{U}\boldsymbol{V}\boldsymbol{U}^{\boldsymbol{T}} = \begin{pmatrix} \boldsymbol{S}^{+} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}^{-} \end{pmatrix} \tag{24}$$

where the $2N \times 2N$ matrix $\boldsymbol{S}$ is defined as

$$\boldsymbol{S}^{\pm} = \begin{pmatrix} S^{\pm}_{11\mathrm{RR}} & S^{\pm}_{11\mathrm{RL}} & S^{\pm}_{12\mathrm{RR}} & S^{\pm}_{12\mathrm{RL}} & \cdots & \cdots & S^{\pm}_{1N\mathrm{RR}} & S^{\pm}_{1N\mathrm{RL}} \\ S^{\pm}_{11\mathrm{RL}} & S^{\pm}_{11\mathrm{RR}} & S^{\pm}_{12\mathrm{RL}} & S^{\pm}_{12\mathrm{RR}} & \cdots & \cdots & S^{\pm}_{1N\mathrm{RL}} & S^{\pm}_{1N\mathrm{RR}} \\ S^{\pm}_{21\mathrm{RR}} & S^{\pm}_{21\mathrm{RL}} & S^{\pm}_{22\mathrm{RR}} & S^{\pm}_{22\mathrm{RL}} & \cdots & \cdots & S^{\pm}_{2N\mathrm{RR}} & S^{\pm}_{2N\mathrm{RLp}} \\ S^{\pm}_{21\mathrm{RL}} & S^{\pm}_{21\mathrm{RR}} & S^{\pm}_{22\mathrm{RL}} & S^{\pm}_{22\mathrm{RR}} & \cdots & \cdots & S^{\pm}_{2N\mathrm{RL}} & S^{\pm}_{2N\mathrm{RR}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S^{\pm}_{N1\mathrm{RR}} & S^{\pm}_{N1\mathrm{RL}} & S^{\pm}_{N2\mathrm{RL}} & S^{\pm}_{N2\mathrm{RR}} & \cdots & \cdots & S^{\pm}_{NN\mathrm{RR}} & S^{\pm}_{NN\mathrm{RL}} \\ S^{\pm}_{N1\mathrm{RL}} & S^{\pm}_{N1\mathrm{RR}} & S^{\pm}_{N2\mathrm{RR}} & S^{\pm}_{N2\mathrm{RL}} & \cdots & \cdots & S^{\pm}_{NN\mathrm{RL}} & S^{\pm}_{NN\mathrm{RR}} \end{pmatrix} \tag{25}$$

with

$$S^{\pm}_{ij\mathrm{RR}} = V_{ij\mathrm{RRp}} \pm V_{ij\mathrm{RRv}} \qquad S^{\pm}_{ij\mathrm{RL}} = V_{ij\mathrm{RLp}} \pm V_{ij\mathrm{RLv}}.$$

So the effective action (21) is given by

$$\begin{aligned} &S_{\mathrm{eff}}[\rho, V] = S_\rho + S_\chi \\ &S_\rho = -\frac{1}{TL}\sum_{q,\omega} \xi^{\mathrm{T}}(-q,-\omega)\boldsymbol{I}^{+}\xi(q,\omega) \\ &S_\chi = -\frac{1}{TL}\sum_{q,\omega} \zeta^{\mathrm{T}}(-q,-\omega)\boldsymbol{I}^{-}\zeta(q,\omega) \end{aligned} \tag{26}$$

with $\boldsymbol{I}^{\pm} = \boldsymbol{S}^{\pm} - \boldsymbol{J}$. As the expression of M_0, $\boldsymbol{J}$ is a $2N \times 2N$ matrix

$$\boldsymbol{J} = \begin{pmatrix} F_{\mathrm{R}} & 0 \\ 0 & F_{\mathrm{L}} \end{pmatrix} \bigotimes \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix}$$

and the vectors in equation (26) are the charge part and the spin part denoted as

$$\begin{aligned} \xi^{\mathrm{T}} &= (\rho_{1\mathrm{R}}, \rho_{1\mathrm{L}}, \rho_{2\mathrm{R}}, \rho_{2\mathrm{L}}, \ldots, \rho_{N\mathrm{R}}, \rho_{N\mathrm{L}}) \\ \zeta^{\mathrm{T}} &= (\chi_{1\mathrm{R}}, \chi_{1\mathrm{L}}, \chi_{2\mathrm{R}}, \chi_{2\mathrm{L}}, \ldots, \chi_{N\mathrm{R}}, \chi_{N\mathrm{L}}). \end{aligned}$$

Obviously the charge and spin have been separated from each other. The system is divided into charge channels and spin channels. In general, spin channels contribute less to the physical properties than charge channels. So in [15], the contribution of the spin channels are cut down, but in our case we still maintain this part. The spectrum of the plasmon is determined by $\det \boldsymbol{I}^{(\pm)}\,|_{\omega=\boldsymbol{E}} = \boldsymbol{0}$, which, generally, give N positive roots for the charge part and the spin part respectively. We denote these roots as $E^{\rho}_i(q)$ and $E^{\chi}_i(q)$, $(i = 1, 2, \ldots, N)$, so the partition function (5) becomes

$$\begin{aligned} \frac{Z}{Z_0} &= \frac{\int [D\rho] \exp[-\mathrm{i}\int \xi^{\mathrm{T}}(-q,-\omega)\boldsymbol{I}^{(+)}\xi(q,\omega)]}{\int [D\rho] \exp[-\mathrm{i}\int \xi^{\mathrm{T}}(-q,-\omega)\boldsymbol{J}\xi(q,\omega)]} \\ &\quad \times \frac{\int [D\chi] \exp[-\mathrm{i}\int \zeta^{\mathrm{T}}(-q,-\omega)\boldsymbol{I}^{(-)}\zeta(q,\omega)]}{\int [D\chi] \exp[-\mathrm{i}\int \zeta^{\mathrm{T}}(-q,-\omega)\boldsymbol{J}\zeta(q,\omega)]} \\ &= \prod_{q,\omega}\left[\frac{\det \boldsymbol{J}(\boldsymbol{q},\omega)}{\det \boldsymbol{I}^{(+)}(\boldsymbol{q},\omega)}\right]^{1/2} \prod_{q,\omega}\left[\frac{\det \boldsymbol{J}(\boldsymbol{q},\omega)}{\det \boldsymbol{I}^{(-)}(\boldsymbol{q},\omega)}\right]^{1/2} \\ &= \exp[-F(T)/T] \end{aligned} \tag{27}$$

where the free energy $F(T)$ for the multi-coupled spin chain system is

$$\begin{aligned} F(T) &= T\sum_{q,\omega}\left[\frac{1}{2}\ln\frac{\det \boldsymbol{I}^{(+)}(\boldsymbol{q},\omega)}{\det \boldsymbol{J}(\boldsymbol{q},\omega)}+\frac{1}{2}\ln\frac{\det \boldsymbol{I}^{(-)}(\boldsymbol{q},\omega)}{\det \boldsymbol{J}(\boldsymbol{q},\omega)}\right] \\ &= T\sum_{q,\omega}\left[\frac{1}{2}\ln\frac{\prod_{i=1}^{N}(\omega^2+E_i^{\rho\,2})}{(\omega^2+q^2)^N}+\frac{1}{2}\ln\frac{\prod_{i=1}^{N}(\omega^2+E_i^{\chi\,2})}{(\omega^2+q^2)^N}\right] \\ &= T\sum_{q,\omega}\left[\frac{1}{2}\sum_{i=1}^{N}\ln\left(1+\frac{E_i^{\rho}(q)^2-q^2}{\omega^2+q^2}\right)+\frac{1}{2}\sum_{i=1}^{N}\ln\left(1+\frac{E_i^{\chi}(q)^2-q^2}{\omega^2+q^2}\right)\right]. \end{aligned} \tag{28}$$

The frequency of the boson is $\omega_n = 2n\pi/\beta$. We can first calculate the summation for the frequency, which gives

$$\begin{aligned} F(T) = F_0(T) &+ \sum_{q>0}\sum_{i=1}^{N}[E_i^{\rho}(q)+E_i^{\chi}(q)-2q] \\ &+2T\sum_{q>0}\sum_{i=1}^{N}\left[\ln\frac{1-\exp(-E_i^{\rho}(q)/T)}{1-\exp(-q/T)}+\ln\frac{1-\exp(-E_i^{\chi}(q)/T)}{1-\exp(-q/T)}\right]. \end{aligned} \tag{29}$$

It is readily proved that our result coincides with ones in [9] for one spin chain or one spinless chain.

4. Green function for the coupled spin chain

The Green function for the system is given by

$$\mathrm{i}G_{\mathrm{R/L}i\sigma}(x,t,x',t') = \frac{\int[D\rho]\bar{\psi}_{\mathrm{R/L}i\sigma}(x,t)\psi_{\mathrm{R/L}i\sigma}(x',t')\exp(-\mathrm{i}S_{\mathrm{eff}}[\rho,V])}{\int[D\rho]\exp(-\mathrm{i}S_{\mathrm{eff}}[\rho,V])}. \tag{30}$$

Using the usual bosonization method [1, 3], the relations between Fermi fields and the boson fields are

$$\psi_{\mathrm{R/L}i\sigma}(x,t) = \left(\frac{1}{2\pi\alpha}\right)^{1/2}\exp[-\mathrm{i}\phi_{\mathrm{R/L}i\sigma}(x,t)]. \tag{31}$$

Substituting equation (31) into (30), we have

$$\mathrm{i}G_{\mathrm{R/L}i\sigma}(x,t,x',t') = \frac{\int[D\rho]\exp[\mathrm{i}(\phi_{\mathrm{R/L}i\sigma}(x,t)-\phi_{\mathrm{R/L}i\sigma}(x',t')-S_{\mathrm{eff}}(\rho,V))]}{\int[D\rho]\exp(-\mathrm{i}S_{\mathrm{eff}}[\rho,V])} \tag{32}$$

where

$$\begin{aligned} \phi_{\mathrm{R/L}i\sigma}(x,t)-\phi_{\mathrm{R/L}i\sigma}(x',t') &= \frac{1}{TL}\sum_{q,\omega}f_{\mathrm{R/L}}(-q,-\omega,x,t,x',t')\rho_{\mathrm{R/L}i\sigma}(q,\omega) \\ f_{\mathrm{R/L}}(-q,-\omega,x,t,x',t') &= \mp\frac{2\pi}{q}[\exp(\mathrm{i}(qx-\omega t))-\exp(\mathrm{i}(qx'-\omega t'))]. \end{aligned} \tag{33}$$

By making use of the transformation (22), Green function (32) is separated into two parts, one for the charge and the other for the spin:

$$\mathrm{i}G_{\mathrm{R/L}i\sigma}(x,t,x',t') = G_{\mathrm{R/L}i}^{(\rho\sigma)}(x,t,x',t')G_{\mathrm{R/L}i}^{(\chi\sigma)}(x,t,x',t') \tag{34}$$

where

$$\begin{aligned} G_{\mathrm{R/L}i}^{(\rho\sigma)}(x,t,x',t') &= \frac{\int[D\rho]\exp[(-)^{\sigma}\mathrm{i}(\phi_{\mathrm{R/L}i}^{(\rho)}(x,t)-\phi_{\mathrm{R/L}i}^{(\rho)}(x',t'))-\mathrm{i}S_{\rho}]}{\int[D\rho]\exp(-\mathrm{i}S_{\rho})} \\ G_{\mathrm{R/L}i}^{(\chi\sigma)}(x,t,x',t') &= \frac{\int[D\chi]\exp[(-)^{\sigma}\mathrm{i}(\phi_{\mathrm{R/L}i}^{(\chi)}(x,t)-\phi_{\mathrm{R/L}i}^{(\chi)}(x',t'))-\mathrm{i}S_{\chi}]}{\int[D\chi]\exp(-\mathrm{i}S_{\chi})}. \end{aligned} \tag{35}$$

When spin is up (down), the symbol σ is chosen as 1 (-1) and the functions above are

$$\begin{aligned}\phi^{(\rho)}_{\mathrm{R/L}i}(x,t)-\phi^{(\rho)}_{\mathrm{R/L}i\sigma}(x',t')&=\frac{1}{\sqrt{2}TL}\sum_{q,\omega}f_{\mathrm{R/L}}(-q,-\omega,x,t,x',t')\rho_{\mathrm{R/L}i}(q,\omega)\\ \phi^{(\chi)}_{\mathrm{R/L}i}(x,t)-\phi^{(\chi)}_{\mathrm{R/L}i}(x',t')&=\frac{1}{\sqrt{2}TL}\sum_{q,\omega}f_{\mathrm{R/L}}(-q,-\omega,x,t,x',t')\chi_{\mathrm{R/L}i}(q,\omega).\end{aligned}\tag{36}$$

Due to the separation of charge and spin parts, the interaction in every effective action is expressed as a $2N\times 2N$ matrix form. It has been shown in equation (26) that the $4N$-interaction-field system reduced to two independent $2N$-interaction-field systems. In some respect, our process simplifies the problem. After integrating, equation (35) gives

$$\begin{aligned}G^{(\rho\sigma)}_{\mathrm{R/L}i}(x,t,x',t')&=\exp[\tfrac{1}{2}P^{(\rho)}_{\mathrm{R/L}i}(x,t,x',t')]\\ G^{(\chi\sigma)}_{\mathrm{R/L}i}(x,t,x',t')&=\exp[\tfrac{1}{2}P^{(\chi)}_{\mathrm{R/L}i}(x,t,x',t')]\end{aligned}\tag{37}$$

where $(i=1,2,\ldots,N)$

$$\begin{aligned}P^{(\rho)}_{\mathrm{R}i}(x,t,x',t')&=\frac{\mathrm{i}}{4TL}\sum_{q,\omega}|f_{\mathrm{R}}(q,\omega,x,t,x',t')|^2\boldsymbol{I}^{(+)^{-1}}_{2i-1,2i-1}(q,\omega)\\ P^{(\rho)}_{\mathrm{L}i}(x,t,x',t')&=\frac{\mathrm{i}}{4TL}\sum_{q,\omega}|f_{\mathrm{L}}(q,\omega,x,t,x',t')|^2\boldsymbol{I}^{(+)^{-1}}_{2i,2i}(q,\omega)\\ P^{(\chi)}_{\mathrm{R}i}(x,t,x',t')&=\frac{\mathrm{i}}{4TL}\sum_{q,\omega}|f_{\mathrm{R}}(q,\omega,x,t,x',t')|^2\boldsymbol{I}^{(-)^{-1}}_{2N+2i-1,2N+2i-1}(q,\omega)\\ P^{(\chi)}_{\mathrm{L}i}(x,t,x',t')&=\frac{\mathrm{i}}{4TL}\sum_{q,\omega}|f_{\mathrm{L}}(q,\omega,x,t,x',t')|^2\boldsymbol{I}^{(-)^{-1}}_{2N+2i,2N+2i}(q,\omega).\end{aligned}\tag{38}$$

At last we see that Green function of the system can be written as

$$\mathrm{i}G_{\mathrm{R/L}i\sigma}(x,t,x',t')=\exp[P^{\rho}_{\mathrm{R/L}i}(x,t,x',t')+P^{\chi}_{\mathrm{R/L}i}(x,t,x',t')].\tag{39}$$

Equation (39) shows that the Green function of the system does not depend on the spin index, because there is no magnetic interaction and the spin in different directions have the same role in the system. The calculation of Green functions contributes to the calculation of elements of the inverse matrix $(\boldsymbol{I}^{(\pm))^{-1}}$. According to the definition of the matrix $\boldsymbol{I}^{(\pm)}$, its inverse matrix has the same symmetry property as the matrix $S^{\pm}$. If a single spin chain is considered, our result in equation (39) coincides with [9]. Although it is difficult to give the elements of the inverse matrix for a multi-coupled chain, theoretically, we can use this formula to give Green functions. It must be noted that due to the symmetry of matrix $\boldsymbol{S}^{\pm}$ we can use another unitary transformation as in equation (23) to transform the vector ρ, χ and the matrix $\boldsymbol{S}^{\pm}$, and further and further, until the matrix $\boldsymbol{I}^{\pm}$ becomes a 2×2 matrix, if the number of coupled chains is $N=2^m$. This process makes the system reduce to a series of decoupled single spinless chains at last. Since the Green function for a single spinless chain is known, the Green function for the multi-coupled spin chain can be solved by this process. However, if the number of coupled chains does not satisfy $N=2^m$, the system is not decoupled to the single spinless chain and we need to solve the eigenvalues of a matrix at last. In section 6, we adopt this method to give the Green function for the two-coupled spin chain.

5. Free energy for the two-coupled spin chain

In order to see the applicability, we apply this method to the two-coupled spin chain. To see it clearly, the interaction potential in this model is given again:

$$\boldsymbol{V_2} = \begin{pmatrix} V_{11\mathrm{RRp}} & V_{11\mathrm{RRv}} & V_{12\mathrm{RRp}} & V_{12\mathrm{RRv}} & V_{11\mathrm{RLp}} & V_{11\mathrm{RLv}} & V_{12\mathrm{RLp}} & V_{12\mathrm{RLv}} \\ V_{11\mathrm{RRv}} & V_{11\mathrm{RRp}} & V_{12\mathrm{RRv}} & V_{12\mathrm{RRp}} & V_{11\mathrm{RLv}} & V_{11\mathrm{RLp}} & V_{12\mathrm{RLv}} & V_{12\mathrm{RLp}} \\ V_{21\mathrm{RRp}} & V_{21\mathrm{RRv}} & V_{22\mathrm{RRp}} & V_{22\mathrm{RRv}} & V_{21\mathrm{RLp}} & V_{21\mathrm{RLv}} & V_{22\mathrm{RLp}} & V_{22\mathrm{RLv}} \\ V_{21\mathrm{RRv}} & V_{21\mathrm{RRp}} & V_{22\mathrm{RRv}} & V_{22\mathrm{RRp}} & V_{21\mathrm{RLv}} & V_{21\mathrm{RLp}} & V_{22\mathrm{RLv}} & V_{22\mathrm{RLp}} \\ V_{11\mathrm{RLp}} & V_{11\mathrm{RLv}} & V_{12\mathrm{RLp}} & V_{12\mathrm{RLv}} & V_{11\mathrm{LLp}} & V_{11\mathrm{LLv}} & V_{12\mathrm{LLp}} & V_{12\mathrm{LLv}} \\ V_{11\mathrm{RLv}} & V_{11\mathrm{RLp}} & V_{12\mathrm{RLv}} & V_{12\mathrm{RLp}} & V_{11\mathrm{LLv}} & V_{11\mathrm{LLp}} & V_{12\mathrm{LLv}} & V_{12\mathrm{LLp}} \\ V_{21\mathrm{RLp}} & V_{21\mathrm{RLv}} & V_{22\mathrm{RLp}} & V_{22\mathrm{RLv}} & V_{21\mathrm{LLp}} & V_{21\mathrm{LLv}} & V_{22\mathrm{LLp}} & V_{22\mathrm{LLv}} \\ V_{21\mathrm{RLv}} & V_{21\mathrm{RLp}} & V_{22\mathrm{RLv}} & V_{22\mathrm{RLp}} & V_{21\mathrm{LLv}} & V_{21\mathrm{LLp}} & V_{22\mathrm{LLv}} & V_{22\mathrm{LLp}} \end{pmatrix}. \tag{40}$$

After integrating out the electron field, the action becomes

$$S_{\mathrm{eff}}[\phi, \rho, V] = -\frac{1}{4\pi TL}\sum_{q,\omega}\sum_{(i=1,2)(\sigma=\uparrow\downarrow)}\left[\frac{q}{v_{\mathrm{F}}q-\omega}|\phi_{iR\sigma}|^2 + \frac{q}{v_{\mathrm{F}}q+\omega}|\phi_{iL\sigma}|^2\right] + \Phi^T\rho + \rho^T\boldsymbol{V}\rho. \tag{41}$$

Integrating out the boson field ϕ, we have the effective action for the density fields

$$S_{\mathrm{eff}}[\rho, V] = \frac{1}{TL}\sum_{q,\omega}\sum_{(i=1,2)(\sigma=\uparrow\downarrow)}[F_{\mathrm{R}}|\rho_{iR\sigma}|^2 + F_{\mathrm{L}}|\rho_{iL\sigma}|^2] - \rho^T\boldsymbol{V}\rho \tag{42}$$

where parameters $F_{\mathrm{R/L}}$ are the same as in equation (21). For the purpose of giving the free energy for the two-coupled spin chain, we select a unitary transformation $\boldsymbol{U}$ in equation (22)

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{43}$$

After this transformation, the interaction potential becomes

$$\boldsymbol{K}'_U = \begin{pmatrix} \boldsymbol{S}^{(+)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}^{(-)} \end{pmatrix} \tag{44}$$

where $\boldsymbol{S}^{(\pm)}$ is a 4×4 matrix.

$$\boldsymbol{S}^{(\pm)} = \begin{pmatrix} V_{11\mathrm{RRp}} \pm V_{11\mathrm{RRv}} & V_{11\mathrm{RLp}} \pm V_{11\mathrm{RLv}} & V_{12\mathrm{RRp}} \pm V_{12\mathrm{RRv}} & V_{12\mathrm{RLp}} \pm V_{12\mathrm{RLv}} \\ V_{11\mathrm{RLp}} \pm V_{11\mathrm{RLv}} & V_{11\mathrm{LLp}} \pm V_{11\mathrm{LLv}} & V_{12\mathrm{RLp}} \pm V_{12\mathrm{RLv}} & V_{12\mathrm{LLp}} \pm V_{12\mathrm{LLv}} \\ V_{21\mathrm{RRp}} \pm V_{21\mathrm{RRv}} & V_{21\mathrm{RLp}} \pm V_{21\mathrm{RLv}} & V_{22\mathrm{RRp}} \pm V_{22\mathrm{RRv}} & V_{22\mathrm{RLp}} \pm V_{22\mathrm{RLv}} \\ V_{21\mathrm{RLp}} \pm V_{21\mathrm{RLv}} & V_{21\mathrm{LLp}} \pm V_{21\mathrm{LLv}} & V_{22\mathrm{RLp}} \pm V_{22\mathrm{RLv}} & V_{22\mathrm{LLp}} \pm V_{22\mathrm{LLv}} \end{pmatrix}. \tag{45}$$

According to the physical characters, these matrix elements for the same coupled spin chain should satisfy

$$V_{11\mathrm{RRp}} = V_{22\mathrm{RRp}} = V_{11\mathrm{LLp}} = V_{22\mathrm{LLp}}$$

and

$$V_{11\mathrm{RRv}} = V_{22\mathrm{RRv}} = V_{11\mathrm{LLv}} = V_{22\mathrm{LLv}}$$
$$V_{12\mathrm{RRp}} = V_{21\mathrm{RRp}} = V_{12\mathrm{LLp}} = V_{21\mathrm{LLp}}$$
$$V_{12\mathrm{RRv}} = V_{21\mathrm{RRv}} = V_{12\mathrm{LLv}} = V_{21\mathrm{LLv}}$$
$$V_{12\mathrm{RLp}} = V_{21\mathrm{RLp}} \qquad V_{12\mathrm{RLv}} = V_{21\mathrm{LLv}}$$
$$V_{11\mathrm{RLp}} = V_{22\mathrm{RLp}} \qquad V_{11\mathrm{RLv}} = V_{22\mathrm{RLv}}.$$

So there are eight different matrix elements in the interaction potential. Under this transformation, the charge and spin field has the following relations with the original boson fields $\rho_{iR/L\sigma}$:

$$(\rho_{1\mathrm{R}}, \rho_{1\mathrm{L}}, \rho_{2\mathrm{R}}, \rho_{2\mathrm{L}}, \chi_{1\mathrm{R}}, \chi_{1\mathrm{L}}, \chi_{2\mathrm{R}}, \chi_{2\mathrm{L}})^{\mathrm{T}} = \boldsymbol{U}(\rho_{1\mathrm{R}\uparrow}, \rho_{1\mathrm{R}\downarrow}, \rho_{2\mathrm{R}\uparrow}, \rho_{2\mathrm{R}\downarrow}\rho_{1\mathrm{L}\uparrow}, \rho_{1\mathrm{L}\downarrow}\rho_{2\mathrm{L}\uparrow}, \rho_{2\mathrm{L}\downarrow})^{\mathrm{T}}. \tag{46}$$

The effective action for the two-coupled spin chain becomes the new form in which the charge and spin fields are separated

$$S_{\mathrm{eff}}[\rho, V] = S_\rho + S_\chi \tag{47}$$

where the charge part is

$$S_\rho = \frac{1}{TL}\sum_{q,\omega}\sum_i [F_{\mathrm{R}}|\rho_{iR}(q,\omega)|^2 + F_{\mathrm{L}}|\rho_{iL}(q,\omega)|^2] - \frac{1}{TL}\sum_{q,\omega}\rho^{\mathrm{T}}(-q,-\omega)\boldsymbol{S}^{+}\rho(q,\omega) \tag{48}$$

and the spin part is

$$S_\chi = \frac{1}{TL}\sum_{q,\omega}\sum_i [F_{\mathrm{R}}|\chi_{iR}(q,\omega)|^2 + F_{\mathrm{L}}|\chi_{iL}(q,\omega)|^2] - \frac{1}{TL}\sum_{q,\omega}\chi^{\mathrm{T}}(-q,-\omega)\boldsymbol{S}^{-}\chi(q,\omega) \tag{49}$$

with $\rho^{\mathrm{T}} = (\rho_{1\mathrm{R}}, \rho_{1\mathrm{L}}, \rho_{2\mathrm{R}}, \rho_{2\mathrm{L}})$ and $\chi^{\mathrm{T}} = (\chi_{1\mathrm{R}}, \chi_{1\mathrm{L}}, \chi_{2\mathrm{R}}, \chi_{2\mathrm{L}})$. The parameters $F_{\mathrm{R/L}}$ are the same as in equation (21). If the magnetic effect is not considered, it is reasonable for us to assume that the two matrices $\boldsymbol{S}^{\pm}$ have the following forms, which correspond to equation (45) for every element:

$$\boldsymbol{S}^{(\pm)} = \begin{pmatrix} a^{\pm} & a^{\pm} & b^{\pm} & b^{\pm} \\ a^{\pm} & a^{\pm} & b^{\pm} & b^{\pm} \\ b^{\pm} & b^{\pm} & a^{\pm} & a^{\pm} \\ b^{\pm} & b^{\pm} & a^{\pm} & a^{\pm} \end{pmatrix}. \tag{50}$$

So the plasmon spectra are calculated from

$$\det\left[\boldsymbol{S}^{\pm} - \begin{pmatrix} F_{\mathrm{R}} & 0 & 0 & 0 \\ 0 & F_{\mathrm{L}} & 0 & 0 \\ 0 & 0 & F_{\mathrm{R}} & 0 \\ 0 & 0 & 0 & F_{\mathrm{L}} \end{pmatrix}\right]\Bigg|_{\omega=E} = 0.$$

After calculation, the spectra of the system are

$$E_1^{\pm}(q) = |q|\left[v_{\mathrm{F}}^2 + \frac{2a^{\pm}v_{\mathrm{F}}}{\pi} + \frac{2}{\pi^2}\sqrt{b^{\pm 2}(\pi v_{\mathrm{F}} + a^{\pm})^2 - a^{\pm}b^{\pm 2}(2\pi v_{\mathrm{F}} + a^{\pm})}\right]^{1/2} \tag{51}$$

and

$$E_2^{\pm}(q) = |q|\left[v_{\mathrm{F}}^2 + \frac{2a^{\pm}v_{\mathrm{F}}}{\pi} - \frac{2}{\pi^2}\sqrt{b^{\pm 2}(\pi v_{\mathrm{F}} + a^{\pm})^2 - a^{\pm}b^{\pm 2}(2\pi v_{\mathrm{F}} + a^{\pm})}\right]^{1/2}. \tag{52}$$

Lending to equation (29), the free energy for the two-coupled spin chain is

$$F(T) = F_0(T) + \sum_{q>0}\sum_{i=1}^{2}[E_i^+(q) + E_i^-(q) - 2q] \\ +2T\sum_{q>0}\sum_{i=1}^{2}\left[\ln\frac{1-\exp(-E_i^+(q)/T)}{1-\exp(-q/T)} + \ln\frac{1-\exp(-E_i^-(q)/T)}{1-\exp(-q/T)}\right]. \tag{53}$$

6. Green function for two-coupled spin chain

In section 4, we showed our idea to solve the Green function for the multi-coupled spin chain. Now a concrete example for the two-coupled chain is given. In section 5, by a unitary transformation (43), the action for the two-coupled spin chain has been turned into the form (47). It is seen that the charge and spin have been separated and expressed as charge vector ρ and spin vector χ. In the meantime, the Green function has been written as equation (34), but the matrices $\boldsymbol{I}^{\pm}$ in it are 4×4 ones. It is important for us to notice that $\boldsymbol{I}^{\pm}$ still has the same symmetry as $S^{\pm}$, so we can perform a further transformation to reduce the action. Lending to the unitary transformation (43), the next transformation for the effective action (48), (49) is

$$\boldsymbol{U}' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \tag{54}$$

which transforms the two vectors ρ and χ into

$$(\rho_{\mathrm{R}}^{(+)}, \rho_{\mathrm{L}}^{(+)}, \rho_{\mathrm{R}}^{(-)}, \rho_{\mathrm{L}}^{(-)})^{\mathrm{T}} = \boldsymbol{U}'(\rho_{1\mathrm{R}}, \rho_{1\mathrm{L}}, \rho_{2\mathrm{R}}, \rho_{2\mathrm{L}})^{\mathrm{T}} \tag{55}$$

$$(\chi_{\mathrm{R}}^{(+)}, \chi_{\mathrm{L}}^{(+)}, \chi_{\mathrm{R}}^{(-)}, \chi_{\mathrm{L}}^{(-)})^{\mathrm{T}} = \boldsymbol{U}'(\chi_{1\mathrm{R}}, \chi_{1\mathrm{L}}, \chi_{2\mathrm{R}}, \chi_{2\mathrm{L}})^{\mathrm{T}} \tag{56}$$

and the matrix $\boldsymbol{S}^{\pm}$ becomes a diagonal matrix form

$$\boldsymbol{D}^{(\pm)} \equiv \boldsymbol{U}'\boldsymbol{S}^{\pm}\boldsymbol{U}'T = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{1}}^{(\pm)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{\boldsymbol{2}}^{(\pm)} \end{pmatrix} \tag{57}$$

where

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{1}}^{(\pm)} &= \begin{pmatrix} v_1^{(\pm)} & v_2^{(\pm)} \\ v_2^{(\pm)} & v_1^{(\pm)} \end{pmatrix} \\ \boldsymbol{A}_{\boldsymbol{2}}^{(\pm)} &= \begin{pmatrix} \delta_1^{(\pm)} & \delta_2^{(\pm)} \\ \delta_2^{(\pm)} & \delta_1^{(\pm)} \end{pmatrix} \\ v_1^{(\pm)} &= \sum_{i,j}[V_{ij\mathrm{RRp}} \pm V_{ij\mathrm{RRv}}] \\ v_2^{(\pm)} &= \sum_{i,j}[V_{ij\mathrm{RLp}} \pm V_{ij\mathrm{RLv}}] \\ \delta_1^{(\pm)} &= \sum_{i,j}[V_{ii\mathrm{RRp}} \pm V_{ii\mathrm{RRv}}] - \sum_{i\neq j}[V_{ij\mathrm{RRp}} \pm V_{ij\mathrm{RRv}}] \\ \delta_2^{(\pm)} &= \sum_{i}[V_{ii\mathrm{RLp}} \pm V_{ii\mathrm{RLv}}] - \sum_{i\neq j}[V_{ij\mathrm{RLp}} \pm V_{ij\mathrm{RLv}}]. \end{aligned} \tag{58}$$

After the unitary transformation, the effective action (47) becomes the sum of four independent actions

$$S_{\mathrm{eff}}[\rho, V] = S[\rho^{(+)}] + S[\rho^{(-)}] + S[\chi^{(+)}] + S[\chi^{(-)}] \tag{59}$$

where

$$S[\rho^{(+)}] = \frac{1}{TL}\sum_{q,\omega}\sum_{i}[F_{\mathrm{R}}|\rho_{\mathrm{R}}^{(+)}(q,\omega)|^2 + F_{\mathrm{L}}|\rho_{\mathrm{L}}^{(+)}(q,\omega)|^2] - \frac{1}{TL}\sum_{q,\omega}(\rho_{\mathrm{R}}^{(+)}(-q,-\omega),\rho_{\mathrm{L}}^{(+)}(-q,-\omega))\boldsymbol{A_1}^{(+)}\begin{pmatrix}\rho_{\mathrm{R}}^{(+)}(q,\omega)\\ \rho_{\mathrm{L}}^{(+)}(q,\omega)\end{pmatrix} \tag{60}$$

$$S[\rho^{(-)}] = \frac{1}{TL}\sum_{q,\omega}\sum_{i}[F_{\mathrm{R}}|\rho_{\mathrm{R}}^{(-)}(q,\omega)|^2 + F_{\mathrm{L}}|\rho_{\mathrm{L}}^{(-)}(q,\omega)|^2] - \frac{1}{TL}\sum_{q,\omega}(\rho_{\mathrm{R}}^{(-)}(-q,-\omega),\rho_{\mathrm{L}}^{(-)}(-q,-\omega))\boldsymbol{A_2}^{(+)}\begin{pmatrix}\rho_{\mathrm{R}}^{(-)}(q,\omega)\\ \rho_{\mathrm{L}}^{(-)}(q,\omega)\end{pmatrix} \tag{61}$$

$$S[\chi^{(+)}] = \frac{1}{TL}\sum_{q,\omega}\sum_{i}[F_{\mathrm{R}}|\chi_{\mathrm{R}}^{(+)}(q,\omega)|^2 + F_{\mathrm{L}}|\chi_{\mathrm{L}}^{(+)}(q,\omega)|^2] - \frac{1}{TL}\sum_{q,\omega}(\chi_{\mathrm{R}}^{(+)}(-q,-\omega),\chi_{\mathrm{L}}^{(+)}(-q,-\omega))\boldsymbol{A_1}^{(-)}\begin{pmatrix}\chi_{\mathrm{R}}^{(+)}(q,\omega)\\ \chi_{\mathrm{L}}^{(+)}(q,\omega)\end{pmatrix} \tag{62}$$

$$S[\chi^{(-)}] = \frac{1}{TL}\sum_{q,\omega}\sum_{i}[F_{\mathrm{R}}|\chi_{\mathrm{R}}^{(-)}(q,\omega)|^2 + F_{\mathrm{L}}|\chi_{\mathrm{L}}^{(-)}(q,\omega)|^2] - \frac{1}{TL}\sum_{q,\omega}(\chi_{\mathrm{R}}^{(-)}(-q,-\omega),\chi_{\mathrm{L}}^{(-)}(-q,-\omega))\boldsymbol{A_2}^{(+)}\begin{pmatrix}\chi_{\mathrm{R}}^{(-)}(q,\omega)\\ \chi_{\mathrm{L}}^{(-)}(q,\omega)\end{pmatrix}. \tag{63}$$

It is seen that these actions (60)–(63) describe the single spinless chains, but they correspond to different physical quantities. $S[\rho^{(+)}]$ corresponds to the total charge moving at the right-hand Fermi point; $S[\rho^{(-)}]$ corresponds to the total charge moving at the left-hand Fermi point, while $S[\chi^{(+)}]$ corresponds to the total spin moving at the right-hand Fermi point and $S[\chi^{(-)}]$ corresponds to the total spin moving at the left-hand Fermi point. All of these are spinless Tomonaga–Luttinger liquid models. Similarly to the result in [9], functions $P_{\mathrm{R/L}}^{\rho^{(+)}}$ in equation (37) for the action $S[\rho^{(+)}]$ can be calculated

$$P_{\mathrm{R/L}}^{\rho^{(+)}}(x,t,x',t') = -\ln\frac{x \mp \bar{v}_{\mathrm{F}}[\rho^{(+)}]t + \mathrm{i}\alpha}{x \mp v_{\mathrm{F}} + \mathrm{i}\alpha} - \mu[\rho^{(+)}]\ln[(x - \bar{v}_{\mathrm{F}}[\rho^{(+)}]t + \mathrm{i}\alpha)(x + \bar{v}_{\mathrm{F}}[\rho^{(+)}]t - \mathrm{i}\alpha)] \tag{64}$$

where for simplicity we have used x and t to replace $x - x'$ and $t - t'$ in the right-hand side of equation (64). Parameters in equation (64) are defined as

$$\mu[\rho^{(+)}] = \frac{1}{4}\left(g[\rho^{(+)}] + \frac{1}{g[\rho^{(+)}]} - 2\right) \qquad \frac{1}{\beta[\rho^{(+)}]} = \frac{1}{2}\left(g[\rho^{(+)}] + \frac{1}{g[\rho^{(+)}]}\right)$$
$$\beta^2[\rho^{(+)}] = 1 - \left[\frac{v_2^{(+)}}{\pi(v_{\mathrm{F}} + v_1^{(+)})}\right]^2 \qquad \bar{v}_{\mathrm{F}}[\rho^{(+)}] = \beta(v_{\mathrm{F}} + v_1^{(+)}). \tag{65}$$

Other functions $P_{\mathrm{R/L}}^{\rho^{(-)}}$, $P_{\mathrm{R/L}}^{\chi^{(+)}}$ and $P_{\mathrm{R/L}}^{\chi^{(-)}}$ in equation (37) are obtained from $P_{\mathrm{R/L}}^{\rho^{(+)}}$ by replacing $(v_1^{(+)}, v_2^{(+)})$ by $(v_1^{(-)}, v_2^{(-)})$, $(\delta_1^{(+)}, \delta_2^{(+)})$ and $(\delta_1^{(-)}, \delta_2^{(-)})$ in equations (64) and (65) respectively. So the Green functions of the system are

$$\mathrm{i}G_{\mathrm{R/L}i\sigma}(x,t,x',t') = \exp\left[\tfrac{1}{2}(P_{\mathrm{R/L}}^{\rho^{(+)}} + P_{\mathrm{R/L}}^{\rho^{(-)}} + P_{\mathrm{R/L}}^{\chi^{(+)}} + P_{\mathrm{R/L}}^{\chi^{(-)}})\right]. \tag{66}$$

Combining equations (64)–(66), the Green function for the two-coupled spin chain is expressed as

$$\mathrm{i}G_{\mathrm{R/L}i\sigma}(x,t,x',t') = \tfrac{1}{2\pi}\exp(\mathrm{i}p_f x)W$$

$$W = \theta(t)\prod_{\kappa=\rho^{(+)},\rho^{(-)}\chi^{(+)},\chi^{(-)}} \frac{1}{(x \mp \bar{v}_{\mathrm{F}}[\kappa]t + \mathrm{i}\alpha)^{1/2}} \frac{1}{[(x-\bar{v}_{\mathrm{F}}[\kappa]t+\mathrm{i}\alpha)(x+\bar{v}_{\mathrm{F}}[\kappa]t-\mathrm{i}\alpha)]^{\mu[\kappa]/2}}$$
$$+\theta(-t)\{cc\}. \tag{67}$$

From this result, we see that the Green function for the two-coupled spin chain is the multiplying form of Green functions in terms of many single spinless non-coupling chains with different parameters, but these parameters are determined by the coupling system. Furthermore, for the 2^m coupled spinless chains, we may use a series of transformations to reduce it to many non-coupled chains and give properties of the system. However, for other numbers of coupled spinless chains, the problem is not so simple, and we have to solve a series of matrices.

7. Conclusion

Only considering the forward scattering, the free energy and the Green function of the multi-coupled spin chain system are derived by using the functional-integral method in the low-energy scales. We deduce that as the treatment for the spin-coupled chains 2^m coupled spinless chains can be reduced into many independent single spinless Tomonaga–Luttinger liquid models, so the Green function and free energy can be obtained easily, while many other properties of the system can be found, but for other numbers of coupled spinless chains, since we cannot reduce them into the form of a single spinless case at last, we have to solve the elements of an inverse matrix such as $I^{\pm}$.

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References

[1] Haldane F D M 1981 *J. Phys. C: Solid State Phys.* **14** 2585
[2] Sölyom J 1979 *Adv. Phys.* **28** 201
[3] Shankar R 1990 *Int. J. Mod. Phys.* B **4** 2371
[4] Mattis D C 1974 *J. Math. Phys.* **15** 609
Mattis D C 1987 *Phys. Rev.* B **36** 745
[5] Luther A and Peschel I 1974 *Phys. Rev.* B **9** 2911
[6] Mattis D C and Lieb E H 1965 *J. Math. Phys.* **6** 304
[7] Gogolin A O, Nersesyan A A and Tsvelik A M *Bosonization and Strongly Correlated Systems* (Cambridge: Cambridge University Press)
[8] Liu Y L and Ng T K 2000 *Phys. Rev.* B **61** 2911
Liu Y L 1999 *Phys. Rev.* B **59** 7973
Liu Y L 2000 *Phys. Rev.* B **62** 1
[9] Lee D K K and Chen Y 1988 *J. Phys. A: Math. Gen.* **21** 4155
[10] Chen Y, Lee D K K and Luchini M V 1988 *Phys. Rev.* B **38** 8497
[11] Schmeltzer D, Zeyher R and Hanke W 1986 *Phys. Rev.* B **33** 5141
[12] Wu C Q, Huang Q F and Sun X 1995 *Phys. Rev.* B **52** R15683
[13] Luther A and Peschel I 1975 *Phys. Rev.* B **12** 3908

[14] Rice T M, Gopalan S and Sigrist M 1993 *Europhys. Lett.* **23** 445
[15] Egger R 1999 *Phys. Rev. Lett.* **83** 5547
[16] Schwinger J 1962 *Phys. Rev.* **128** 2425